

# PATTERNS OF CONJUNCTIVE FORKS

VAŠEK CHVÁTAL, FRANTIŠEK MATÚŠ AND YORI ZWÓLŠ

**ABSTRACT.** Three events in a probability space form a conjunctive fork if they satisfy specific constraints on conditional independence and covariances. Patterns of conjunctive forks within collections of events are characterized by means of systems of linear equations that have positive solutions. This characterization allows patterns of conjunctive forks to be recognized in polynomial time. Relations to previous work on causal betweenness and on patterns of conditional independence among random variables are discussed.

## 1. MOTIVATION

Hans Reichenbach [19, Chapter 19] defined a *conjunctive fork* as an ordered triple  $(A, B, C)$  of events  $A$ ,  $B$  and  $C$  in a probability space  $(\Omega, \mathcal{F}, P)$  that satisfies

- (1)  $P(AC|B) = P(A|B)P(C|B)$ ,
- (2)  $P(AC|\overline{B}) = P(A|\overline{B})P(C|\overline{B})$ ,
- (3)  $P(A|B) > P(A|\overline{B})$ ,
- (4)  $P(C|B) > P(C|\overline{B})$

where, as usual,  $AC$  is a shorthand for  $A \cap C$  and  $\overline{B}$  denotes the complementary event  $\Omega \setminus B$ . (Readers comparing this definition with Reichenbach's original beware: his notation is modified here by switching the role of  $B$  and  $C$ . To denote the middle event in the fork, he

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used  $C$ , perhaps as mnemonic for ‘common cause’.) Implicit in this definition is the assumption

$$(5) \quad 0 < P(B) < 1,$$

which is needed to define the conditional probabilities in (1)–(4).

A similar notion was introduced earlier in the context of sociology [9, Part I, Section 2], but the context of Reichenbach’s discourse was philosophy of science: conjunctive forks play a central role in his causal theory of time. In this role, they have attracted considerable attention: over one hundred publications, such as [1, 20, 21, 7, 2, 5, 24, 8, 11], refer to them. Yet for all this interest, no one seems to have asked a fundamental question:

*What do ternary relations defined by conjunctive forks look like?*

The purpose of our paper is to answer this question.

An additional stimulus to our work was a previous answer [3] to a similar question,

*What do ternary relations defined by causal betweenness look like?*

Here, *causal betweenness* is another ternary relation on sets of events in probability spaces, also introduced by Reichenbach [19, p. 190] in the context of his causal theory of time. (This relation is reviewed in Section 5.) From this perspective, our paper may be seen as a companion to [3].

## 2. THE MAIN RESULT

Let us write  $(A, B, C)_P$  to signify that  $(A, B, C)$  is a conjunctive fork in a probability space  $(\Omega, \mathcal{F}, P)$  and let us say that events  $A_i$  in this space, indexed by elements  $i$  of a set  $N$ , *fork-represent* a ternary relation  $\mathfrak{r}$  on  $N$  if and only if

$$\mathfrak{r} = \{(i, j, k) \in N^3 : (A_i, A_j, A_k)_P\}.$$

In order to characterize ternary relations on a finite ground set that are fork representable, we need a few definitions.

To begin, call a ternary relation  $\mathfrak{r}$  on a ground set  $N$  a *forkness* if and only if it satisfies

- (6)  $(i, j, i) \in \mathfrak{r} \Rightarrow (j, i, j) \in \mathfrak{r}$
- (7)  $(i, j, i), (j, k, j) \in \mathfrak{r} \Rightarrow (i, k, i) \in \mathfrak{r}$
- (8)  $(i, k, j) \in \mathfrak{r} \Rightarrow (j, k, i) \in \mathfrak{r}$
- (9)  $(i, j, k) \in \mathfrak{r} \Rightarrow (i, j, j), (j, k, k), (k, i, i) \in \mathfrak{r}$
- (10)  $(i, k, j), (i, j, k) \in \mathfrak{r} \Rightarrow (j, k, j) \in \mathfrak{r}$

for all choices  $i, j, k$  in  $N$ .

Given a forkness  $\mathfrak{r}$ , write

$$V_{\mathfrak{r}} = \{i \in N : (i, i, i) \in \mathfrak{r}\}$$

and let  $\overset{\mathfrak{r}}{\sim}$  denote the binary relation defined on  $V_{\mathfrak{r}}$  by

$$i \overset{\mathfrak{r}}{\sim} j \Leftrightarrow (i, j, i) \in \mathfrak{r}.$$

This binary relation is reflexive by definition of  $V_{\mathfrak{r}}$ , it is symmetric by (6), and it is transitive by (7). In short,  $\overset{\mathfrak{r}}{\sim}$  is an equivalence relation. Call a forkness  $\mathfrak{r}$  *regular* if, and only if,

$$(11) \quad (i, j, k) \in \mathfrak{r}, i \overset{\mathfrak{r}}{\sim} i', j \overset{\mathfrak{r}}{\sim} j', k \overset{\mathfrak{r}}{\sim} k' \Rightarrow (i', j', k') \in \mathfrak{r}.$$

The *quotient* of a regular forkness  $\mathfrak{r}$  is the ternary relation whose ground set is the set of equivalence classes of  $\overset{\mathfrak{r}}{\sim}$  and which consists of all triples  $(I, J, K)$  such that  $(i, j, k) \in \mathfrak{r}$  for at least one  $i$  in  $I$ , at least one  $j$  in  $J$ , and at least one  $k$  in  $K$ . (Equivalently, since  $\mathfrak{r}$  is regular,  $(I, J, K)$  belongs to its quotient if and only if  $(i, j, k) \in \mathfrak{r}$  for all  $i$  in  $I$ , for all  $j$  in  $J$ , and for all  $k$  in  $K$ .)

Call a ternary relation  $\mathfrak{q}$  *solvable* if and only if the linear system

$$(12) \quad x_{\{I, K\}} = x_{\{I, J\}} + x_{\{J, K\}} \\ \text{for all } (I, J, K) \text{ in } \mathfrak{q} \text{ with pairwise distinct } I, J, K$$

has a solution with each  $x_{\{I, J\}}$  positive.

**Theorem 1.** *A ternary relation on a finite ground set is fork representable if and only if it is a regular forkness and its quotient is solvable.*

Theorem 1 implies that fork-representability of a ternary relation  $\mathfrak{r}$  on a finite ground set  $N$  can be tested in time polynomial in  $|N|$ . More precisely, polynomial time suffices to test  $\mathfrak{r}$  for being a forkness, for testing its regularity, and for the construction of its quotient  $\mathfrak{q}$ . Solvability of

$\mathfrak{q}$  means solvability of a system of linear equations and linear inequalities, which can be tested in polynomial time by the breakthrough result of [10].

We prove the easier ‘only if’ part of Theorem 1 in Section 3 and we prove the ‘if’ part in Section 4. In Section 5, we comment on causal betweenness and its relationship to causal forks. In the final Section 6, we discuss connections to previous work on patterns of conditional independence.

### 3. PROOF OF THE ‘ONLY IF’ PART

**3.1. REICHENBACH’S DEFINITION RESTATED.** Reichenbach’s definition of a conjunctive fork has a neat paraphrase in terms of random variables. To present it, let us first review a few standard definitions.

The *indicator function*  $\mathbb{1}_E$  of an event  $E$  in a probability space is the random variable defined by  $\mathbb{1}_E(\omega) = 1$  if  $\omega \in E$  and  $\mathbb{1}_E(\omega) = 0$  if  $\omega \in \overline{E}$ . Indicator functions  $\mathbb{1}_A$  and  $\mathbb{1}_C$  are said to be *conditionally independent given*  $\mathbb{1}_B$ , in symbols  $\mathbb{1}_A \perp\!\!\!\perp \mathbb{1}_C | \mathbb{1}_B$ , if and only if events  $A$ ,  $B$ ,  $C$  satisfy (1) and (2). The *covariance* of  $\mathbb{1}_A$  and  $\mathbb{1}_B$ , denoted here as  $\text{cov}(A, B)$ , is defined by

$$\text{cov}(A, B) = P(AB) - P(A)P(B).$$

Since (3) means that  $\text{cov}(A, B) > 0$  and (4) means that  $\text{cov}(B, C) > 0$ , we conclude that

$$(13) \quad (A, B, C)_P \Leftrightarrow \mathbb{1}_A \perp\!\!\!\perp \mathbb{1}_C | \mathbb{1}_B \ \& \ \text{cov}(A, B) > 0 \ \& \ \text{cov}(B, C) > 0.$$

**3.2. A COUPLE OF REICHENBACH’S RESULTS.** Reichenbach [19, p. 160, equation (12)] noted that

$$(14) \quad \mathbb{1}_A \perp\!\!\!\perp \mathbb{1}_C | \mathbb{1}_B \Rightarrow \text{cov}(A, C) = \text{cov}(B, B) \cdot (P(A|B) - P(A|\overline{B})) \cdot (P(C|B) - P(C|\overline{B}))$$

and so [19, p. 158, inequality (1)]

$$(15) \quad (A, B, C)_P \Rightarrow \text{cov}(A, C) > 0.$$

Implication (15) was his reason for calling the fork ‘conjunctive’: it is “a fork which makes the conjunction of the two events more frequent than it would be for independent events” [19, p. 159].

Since

$$\begin{aligned} \text{cov}(B, B)(P(A|B) - P(A|\overline{B})) &= \\ P(AB)(1 - P(B)) - P(A\overline{B})P(B) &= \text{cov}(A, B) \end{aligned}$$

and, similarly,  $\text{cov}(B, B)(P(C|B) - P(C|\overline{B})) = \text{cov}(C, B)$ , Reichenbach's implication (14) can be stated as

$$(16) \quad \mathbb{1}_A \perp\!\!\!\perp \mathbb{1}_C | \mathbb{1}_B \Rightarrow \text{cov}(A, C) \cdot \text{cov}(B, B) = \text{cov}(A, B) \cdot \text{cov}(B, C).$$

**3.3. THE IDEA OF THE PROOF.** An event  $E$  is called *P-nontrivial* if and only if  $0 < P(E) < 1$ , which is equivalent to  $\text{cov}(E, E) > 0$ . When  $E, F$  are *P-nontrivial* events, the *correlation* of their indicator functions, denoted here as  $\text{corr}(E, F)$ , is defined by

$$\text{corr}(E, F) = \frac{\text{cov}(E, F)}{\text{cov}(E, E)^{1/2} \text{cov}(F, F)^{1/2}}.$$

In these terms, (16) reads

$$(17) \quad \mathbb{1}_A \perp\!\!\!\perp \mathbb{1}_C | \mathbb{1}_B \Rightarrow \text{corr}(A, C) = \text{corr}(A, B) \cdot \text{corr}(B, C).$$

The strict inequalities (3), (4), (5) imply that

$$(18) \quad \begin{aligned} &\text{in every conjunctive fork } (A, B, C), \\ &\text{all three events } A, B, C \text{ are } P\text{-nontrivial,} \end{aligned}$$

and so  $\text{corr}(A, B)$ ,  $\text{corr}(A, C)$ ,  $\text{corr}(B, C)$  are well defined; (13) guarantees that  $\text{corr}(A, B) > 0$ ,  $\text{corr}(B, C) > 0$  and (15) guarantees that  $\text{corr}(A, C) > 0$ .

Fact (17) guarantees that the system

$$x_{\{A, C\}} = x_{\{A, B\}} + x_{\{B, C\}} \text{ for all conjunctive forks } (A, B, C)$$

can be solved by setting  $x_{\{E, F\}} = -\ln \text{corr}(E, F)$ . This observation goes a long way toward proving the ‘only if’ part of Theorem 1, but it does not quite get there: For instance, if  $(A, B, C)$  is a conjunctive fork and  $A \stackrel{P}{=} B$ , then  $\text{corr}(A, B) = 1$ , and so  $x_{\{A, B\}} = 0$ , but the ‘only if’ part of the theorem requires  $x_{\{A, B\}} > 0$ . To get around such obstacles, we deal with the quotient of the ternary relation made from conjunctive forks.

3.4. OTHER PRELIMINARIES. Events  $E$  and  $F$  are said to be  $P$ -equal, in symbols  $E \stackrel{P}{=} F$ , if and only if  $P(E \Delta F) = 0$ . We claim that

$$(19) \quad (A, B, A)_P \text{ if and only if } A \text{ is } P\text{-nontrivial and } A \stackrel{P}{=} B,$$

$$(20) \quad \text{if } (A, B, C)_P \text{ and } (A, C, B)_P, \text{ then } B \stackrel{P}{=} C.$$

To justify claim (19), note that  $(A, B, A)_P$  means the conjunction of  $\text{cov}(A, B) > 0$  and at least one of  $P(A) = 0$ ,  $P(A) = 1$ ,  $A \stackrel{P}{=} \overline{B}$ ,  $A \stackrel{P}{=} B$ ; of the four equalities, only the last one is compatible with  $\text{cov}(A, B) > 0$ . To justify claim (20), note that  $\text{cov}(B, B) - \text{cov}(B, C) = P(B)P(\overline{B}C) + P(\overline{B})P(B\overline{C})$ ; when  $B$  is  $P$ -nontrivial, the right-hand side vanishes if and only if  $P(B \Delta C) = 0$ ; it follows that

$$\begin{aligned} &\text{if } (A, B, C)_P, \text{ then } \text{cov}(B, B) \geq \text{cov}(B, C) \\ &\text{with equality if and only if } B \stackrel{P}{=} C; \end{aligned}$$

by (16), this implies that

$$\begin{aligned} &\text{if } (A, B, C)_P, \text{ then } \text{cov}(A, B) \geq \text{cov}(A, C) \\ &\text{with equality if and only if } B \stackrel{P}{=} C, \end{aligned}$$

which in turn implies (20).

### 3.5. THE PROOF.

**Lemma 2.** *A ternary relation is fork representable only if it is a regular forkness.*

*Proof.* Consider a fork-representable ternary relation  $\mathfrak{r}$  on a ground set  $N$ . Proving the lemma means verifying that  $\mathfrak{r}$  has properties (6), (7), (8), (9), (10), (11). Since  $\mathfrak{r}$  is fork representable, there are events  $A_i$  in in some probability space  $(\Omega, \mathcal{F}, P)$ , with  $i$  ranging over  $N$ , such that

$$(i, j, k) \in \mathfrak{r} \Leftrightarrow (A_i, A_j, A_k)_P.$$

Properties (6) and (7),

$$\begin{aligned} (i, j, i) \in \mathfrak{r} &\Rightarrow (j, i, j) \in \mathfrak{r}, \\ (i, j, i), (j, k, j) \in \mathfrak{r} &\Rightarrow (i, k, i) \in \mathfrak{r}, \end{aligned}$$

follow from (19). Property (8),

$$(i, k, j) \in \mathfrak{r} \Rightarrow (j, k, i) \in \mathfrak{r},$$

is implicit in the definition of  $(A_i, A_j, A_k)_P$ . Property (9),

$$(i, j, k) \in \mathfrak{r} \Rightarrow (i, j, j), (j, k, k), (k, i, i) \in \mathfrak{r},$$

follows from (15). Property (10),

$$(i, j, k) \in \mathfrak{r} \Rightarrow (i, j, j), (j, k, k), (k, i, i) \in \mathfrak{r},$$

follows from (20) and (19). Property (11),

$$(i, j, k) \in \mathfrak{r}, i \stackrel{\tau}{\sim} i', j \stackrel{\tau}{\sim} j', k \stackrel{\tau}{\sim} k' \Rightarrow (i', j', k') \in \mathfrak{r},$$

follows from (19) alone.  $\square$

**Lemma 3.** *A regular forkness is fork representable only if its quotient is solvable.*

*Proof.* Consider a fork-representable regular forkness  $\mathfrak{r}$  on a ground set  $N$ . Since  $\mathfrak{r}$  is fork representable, there are events  $A_i$  in some probability space  $(\Omega, \mathcal{F}, P)$ , with  $i$  ranging over  $N$ , such that

$$(i, j, k) \in \mathfrak{r} \Leftrightarrow (A_i, A_j, A_k)_P.$$

With  $\mathfrak{q}$  standing for the quotient of  $\mathfrak{r}$ , proving the lemma means finding positive numbers  $x_{\{I, J\}}$  such that

$$(21) \quad x_{\{I, K\}} = x_{\{I, J\}} + x_{\{J, K\}} \\ \text{for all } (I, J, K) \text{ in } \mathfrak{q} \text{ with pairwise distinct } I, J, K.$$

We claim that this can be done by first choosing an element  $r(I)$  from each equivalence class  $I$  of  $\stackrel{\tau}{\sim}$  and then setting

$$x_{\{I, J\}} = -\ln \text{corr}(A_{r(I)}, A_{r(J)})$$

for every pair  $I, J$  of distinct equivalence classes that appear together in a triple in  $\mathfrak{q}$ . (By (19), the right hand side depends only on  $I$  and  $J$  rather than the choice of  $r(I)$  and  $r(J)$ .)

To justify this claim, note first that (21) is satisfied by virtue of (17). A special case of the *covariance inequality* guarantees that every pair  $A, B$  of events satisfies

$$(\text{cov}(A, B))^2 \leq \text{cov}(A, A) \cdot \text{cov}(B, B)$$

and that the two sides are equal if and only if  $A \stackrel{P}{=} B$  or  $A \stackrel{P}{=} \overline{B}$  or  $P(A) = 0$  or  $P(A) = 1$  or  $P(B) = 0$  or  $P(B) = 1$ . If equivalence classes  $I, J$  of  $\stackrel{\tau}{\sim}$  are distinct, then  $A_{r(I)} \not\stackrel{P}{=} A_{r(J)}$ ; if, in addition,  $I$  and  $J$  appear together in a triple in  $\mathfrak{q}$ , then  $A_{r(I)} \not\stackrel{P}{=} \overline{A_{r(J)}}$  (since  $\text{corr}(A_{r(I)}, A_{r(J)}) > 0$  by (13) and (15)) and  $0 < P(A_{r(I)}) < 1$ ,  $0 < P(A_{r(J)}) < 1$  (by (18)). In this case,

$$\text{cov}(A_{r(I)}, A_{r(J)})^2 < \text{cov}(A_{r(I)}, A_{r(I)}) \cdot \text{cov}(A_{r(J)}, A_{r(J)}),$$

and so  $x_{\{I, J\}} > 0$ .  $\square$

## 4. PROOF OF THE ‘IF’ PART

**Lemma 4.** *If the quotient of a regular forkness on a finite ground set is solvable, then it is fork representable.*

*Proof.* Given the quotient  $\mathfrak{q}$  of a regular forkness on a finite ground set along with positive numbers  $x_{\{I,J\}}$  such that

$$(22) \quad x_{\{I,K\}} = x_{\{I,J\}} + x_{\{J,K\}} \\ \text{for all } (I, J, K) \text{ in } \mathfrak{q} \text{ with pairwise distinct } I, J, K,$$

we have to construct a probability space  $(\Omega, \mathcal{F}, P)$  and events  $A_I$  in this space, indexed by elements  $I$  of the ground set  $C$  of  $\mathfrak{q}$ , such that

$$(23) \quad \mathfrak{q} = \{(I, J, K) \in C^3 : (A_I, A_J, A_K)_P\}.$$

For this purpose, we let  $\Omega$  be the power set  $2^C$  of  $C$ , we let  $\mathcal{F}$  be the power set  $2^\Omega$  of  $\Omega$ , and we set

$$A_I = \{\omega \in \Omega : I \in \omega\}.$$

Now let us construct the probability measure  $P$ . Set  $n = |C|$ . Given a subset  $L$  of  $C$ , consider the function  $\chi_L : \Omega \rightarrow \{+1, -1\}$  defined by

$$\chi_L(\omega) = (-1)^{|\omega \cap L|}.$$

Let  $E_{\mathfrak{q}}$  stand for the family of all two-element subsets  $\{I, J\}$  of  $C$  such that  $I$  and  $J$  appear together in a triple in  $\mathfrak{q}$ . Let  $M_{\mathfrak{q}}$  stand for the family of all three-element subsets  $\{I, J, K\}$  of  $C$  such that  $\{I, J\}$ ,  $\{J, K\}$ , and  $\{K, I\}$  belong to  $E_{\mathfrak{q}}$  and no triple in  $\mathfrak{q}$  is formed by all three  $I, J, K$ . Finally, for positive numbers  $\gamma$  and  $\varepsilon$  that are sufficiently small in a sense to be specified shortly, define  $P : \Omega \rightarrow \mathbf{R}$  by

$$P(\omega) = 2^{-n} \left[ 1 + \sum_{\{I,J\} \in E_{\mathfrak{q}}} \chi_{\{I,J\}}(\omega) \gamma^{x_{\{I,J\}}} + \varepsilon \sum_{\{I,J,K\} \in M_{\mathfrak{q}}} \chi_{\{I,J,K\}}(\omega) \right].$$

(In exponents, we write  $x_{\{I,J\}}$  in place of  $x_{\{I,J\}.}$ ) Here, readers with background in harmonic analysis will have recognized the Fourier-Stieltjes transform on the group  $\mathbb{Z}_2$  and the characters  $\chi_L$ .

When  $L$  is nonempty,  $\chi_L$  takes each of the values  $\pm 1$  on the same number its arguments  $\omega$ , and so  $\sum_{\omega \in \Omega} \chi_L(\omega) = 0$ , which implies that  $\sum_{\omega \in \Omega} P(\omega) = 1$ . If  $\gamma < n^{-2/x_{\{I,J\}}}$  for all  $\{I, J\} \in E_{\mathfrak{q}}$  and  $\varepsilon < n^{-3}$ , then

$$2^n P(\omega) \geq 1 - |E_{\mathfrak{q}}| n^{-2} - |M_{\mathfrak{q}}| n^{-3} > 0,$$

so that  $P$  is a probability measure, positive on the elementary events.



The bulk of the proof consists of verifying that this construction satisfies (23). We may assume that  $C \neq \emptyset$  (else  $\Omega$  is a singleton, and so (23) holds). Given a subset  $S$  of  $C$ , write

$$A_S = \{\omega \in \Omega : S \subseteq \omega\}.$$

Since

$$\sum_{\omega \in A_S} \chi_L(\omega) = \begin{cases} (-1)^{|L|} 2^{n-|S|} & \text{if } L \subseteq S, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$(24) \quad 2^{|S|} P(A_S) = 1 + \sum_{\{I, J\} \in E_q, I, J \in S} \gamma^{x\{I, J\}} - \varepsilon \left| \{ \{I, J, K\} \in M_q : I, J, K \in S \} \right|.$$

In particular, formula (24) yields for all  $I$  in  $C$

$$P(A_I) = \frac{1}{2}$$

and it yields for all choices of distinct  $I, J$  in  $C$

$$(25) \quad P(A_{\{I, J\}}) = \frac{1}{4} + \begin{cases} \frac{1}{4} \gamma^{x\{I, J\}} & \text{if } \{I, J\} \in E_q, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$(26) \quad \text{cov}(A_I, A_J) = \begin{cases} \frac{1}{4} \gamma^{x(\{I, J\})} & \text{if } \{I, J\} \in E_q, \\ 0 & \text{otherwise.} \end{cases}$$

For future reference, note also that, by definition,

$$(27) \quad \mathbf{q} \text{ is a forkness such that } (I, I, I) \in \mathbf{q} \text{ for all } I \text{ in } C \\ \text{and such that } (I, J, I) \notin \mathbf{q} \text{ whenever } I \neq J.$$

and that

$$(28) \quad \{I, J\} \in E_q \Leftrightarrow (I, J, J) \in \mathbf{q}.$$

(here, implication  $\Rightarrow$  follows from properties (8) and (9) of forkness and implication  $\Leftarrow$  follows straight from the definition of  $E_q$ ).

Now we are ready to verify (23). Given a triple  $(I, J, K)$  in  $C^3$ , we have to show that

$$(29) \quad (A_I, A_J, A_K)_P \Leftrightarrow (I, J, K) \in \mathbf{q}.$$

CASE 1:  $I = J = K$ .

Since  $C \neq \emptyset$ , all  $I$  in  $C$  are  $P$ -nontrivial, and so we have  $(A_I, A_I, A_I)_P$ . By (27), we have  $(I, I, I) \in \mathbf{q}$ .

CASE 2:  $I \neq J, K = I$ .

Here,  $A_I \not\stackrel{P}{=} A_J$ , and so (19) implies that  $(A_I, A_J, A_I)$  is not a conjunctive fork. By (27), we have  $(I, J, I) \notin \mathfrak{q}$ .

CASE 3:  $I \neq J, K = J$ .

If  $\{I, J\} \in E_{\mathfrak{q}}$ , then (26) guarantees  $\text{cov}(A_I, A_J) > 0$ , which implies  $(A_I, A_J, A_J)_P$ . By (28), we have  $(I, J, J) \in \mathfrak{q}$ .

If  $\{I, J\} \notin E_{\mathfrak{q}}$ , then (26) guarantees  $\text{cov}(A_I, A_J) = 0$ , and so  $(A_I, A_J, A_J)$  is not a conjunctive fork. By (28), we have  $(I, J, J) \notin \mathfrak{q}$ .

CASE 4:  $I = J, K \neq J$ .

This case is reduced to CASE 3 by the flip  $I \leftrightarrow K$ , which preserves both sides of (29).

CASE 5:  $I, J, K$  are pairwise distinct and at least one of  $\{I, J\}, \{J, K\}, \{K, I\}$  does not belong to  $E_{\mathfrak{q}}$ .

By (26), at least one of the covariances  $\text{cov}(A_I, A_J)$ ,  $\text{cov}(A_J, A_K)$ ,  $\text{cov}(A_K, A_I)$  vanishes, and so (13), (15) guarantee that  $(A_I, A_J, A_K)$  is not a conjunctive fork. By definition of  $E_{\mathfrak{q}}$ , we have  $(I, J, K) \notin \mathfrak{q}$ .

CASE 6:  $I, J, K$  are pairwise distinct and all of  $\{I, J\}, \{J, K\}, \{K, I\}$  belong to  $E_{\mathfrak{q}}$ .

By (26), all of  $\text{cov}(A_I, A_J)$ ,  $\text{cov}(A_J, A_K)$ ,  $\text{cov}(A_K, A_I)$  are positive. Now (13) implies that  $(A_I, A_J, A_K)_P$  is equivalent to  $\mathbb{1}_{A_I} \perp \mathbb{1}_{A_K} | \mathbb{1}_{A_J}$ , which means the conjunction of

$$\begin{aligned} \frac{1}{2}P(A_{\{I,J,K\}}) &= P(A_{\{I,J\}})P(A_{\{J,K\}}), \\ \frac{1}{2}[P(A_{\{I,K\}}) - P(A_{\{I,J,K\}})] &= [\frac{1}{2} - P(A_{\{I,J\}})] \cdot [\frac{1}{2} - P(A_{\{J,K\}})]. \end{aligned}$$

Substitution from (25) converts these two equalities to

$$(30) \quad 8P(A_{\{I,J,K\}}) = 1 + \gamma^{x_{\{I,J\}}} + \gamma^{x_{\{J,K\}}} + \gamma^{x_{\{I,J\}} + x_{\{J,K\}}}$$

$$(31) \quad 8P(A_{\{I,J,K\}}) = 1 + \gamma^{x_{\{I,J\}}} + \gamma^{x_{\{J,K\}}} - \gamma^{x_{\{I,J\}} + x_{\{J,K\}}} + 2\gamma^{x_{\{I,K\}}}.$$

Conjunction of (30) and (31) is equivalent to the conjunction of (30) and

$$(32) \quad x_{\{I,J\}} + x_{\{J,K\}} = x_{\{I,K\}}.$$

To summarize,  $(A_I, A_J, A_K)_P$  is equivalent to the conjunction of (30) and (32).

SUBCASE 6.1:  $\{I, J, K\} \in M_{\mathfrak{q}}$ .

In this subcase, formula (24) yields

$$8P(A_{\{I,J,K\}}) = 1 + \gamma^{x_{\{I,J\}}} + \gamma^{x_{\{J,K\}}} + \gamma^{x_{\{I,K\}}} - \varepsilon,$$

which reduces (30) to  $\gamma^{x\{I,K\}-\varepsilon} = \gamma^{x\{I,J\}+x\{J,K\}}$ . This is inconsistent with (32), and so  $(A_I, A_J, A_K)$  is not a conjunctive fork. By definition of  $M_{\mathfrak{q}}$ , we have  $(I, J, K) \notin \mathfrak{q}$ .

**SUBCASE 6.2:**  $\{I, J, K\} \notin M_{\mathfrak{q}}$ .

In this subcase, formula (24) yields

$$8P(A_{\{I,J,K\}}) = 1 + \gamma^{x\{I,J\}} + \gamma^{x\{J,K\}} + \gamma^{x\{I,K\}},$$

which reduces (30) to (32), and so  $(A_I, A_J, A_K)_P$  is equivalent to (32) alone. Now completing the proof means verifying that

$$x_{\{I,J\}} + x_{\{J,K\}} = x_{\{I,K\}} \Leftrightarrow (I, J, K) \in \mathfrak{q}.$$

Implication  $\Leftarrow$  is (22). To prove the reverse implication, note first that by (22) along with  $x_{\{I,J\}} > 0$  and  $x_{\{J,K\}} > 0$ , we have

$$(33) \quad x_{\{I,J\}} + x_{\{J,K\}} = x_{\{I,K\}} \Rightarrow x_{\{J,K\}} < x_{\{I,K\}} \Rightarrow (J, I, K) \notin \mathfrak{q},$$

$$(34) \quad x_{\{I,J\}} + x_{\{J,K\}} = x_{\{I,K\}} \Rightarrow x_{\{I,J\}} < x_{\{I,K\}} \Rightarrow (I, K, J) \notin \mathfrak{q}.$$

By assumptions of this case and subcase, some triple in  $\mathfrak{q}$  is formed by all three  $I, J, K$  and so, since  $\mathfrak{q}$  is a forkness, (8) with  $\mathfrak{q}$  in place of  $\mathfrak{r}$  guarantees that at least one of  $(J, I, K)$ ,  $(I, K, J)$ ,  $(I, J, K)$  belongs to  $\mathfrak{q}$ . If  $x_{\{I,J\}} + x_{\{J,K\}} = x_{\{I,K\}}$ , then (33) and (34) exclude the first two options, and so we have  $(I, J, K) \in \mathfrak{q}$ .  $\square$

**Lemma 5.** *If a regular forkness has a fork-representable quotient, then it is fork representable.*

*Proof.* Given a regular forkness  $\mathfrak{r}$ , a probability space  $(\Omega^0, \mathcal{F}^0, P^0)$ , and events  $A_I^0$  in this space, indexed by elements  $I$  of the ground set  $C$  of the quotient  $\mathfrak{q}$  of  $\mathfrak{r}$ , such that

$$\mathfrak{q} = \{(I, J, K) \in C^3 : (A_I^0, A_J^0, A_K^0)_{P^0}\},$$

we have to construct a probability space  $(\Omega, \mathcal{F}, P)$  and events  $A_i$  in this space, indexed by elements  $i$  of the ground set  $N$  of  $\mathfrak{r}$ , such that

$$(35) \quad \mathfrak{r} = \{(i, j, k) \in N^3 : (A_i, A_j, A_k)_P\}.$$

For this purpose, we let  $\Omega$  be the power set  $2^N$  of  $N$ , we let  $\mathcal{F}$  be the power set  $2^\Omega$  of  $\Omega$ , and we set

$$A_i = \{\omega \in \Omega : i \in \omega\}.$$

For each element  $\omega$  of  $\Omega$  such that every equivalence class  $I$  of  $\sim$  satisfies  $I \subseteq \omega$  or  $I \cap \omega = \emptyset$ , define  $P(\omega) = P^0(\omega^0)$ , where  $\omega^0$  is the set of equivalence classes of  $\sim$  contained in  $\omega$ . For all other elements  $\omega$  of  $\Omega$ , define  $P(\omega) = 0$ . Now verifying (35) is a routine matter.  $\square$

## 5. CAUSAL BETWEENNESS

Reichenbach [19, p. 190] defined an event  $B$  to be *causally between* events  $A$  and  $C$  if

$$\begin{aligned} 1 &> P(A|B) > P(A|C) > P(A) > 0, \\ 1 &> P(C|B) > P(C|A) > P(C) > 0, \\ P(C|AB) &= P(C|B). \end{aligned}$$

Implicit in this definition is the assumption  $P(B) > 0$  that makes  $P(A|B)$  and  $P(C|B)$  meaningful. In turn,  $P(A|B) > 0$  means  $P(AB) > 0$ , which makes  $P(C|AB)$  meaningful. If  $B$  is causally between  $A$  and  $C$ , then all three events are  $P$ -nontrivial and no two of them  $P$ -equal.

If  $(A, B, C)$  is a conjunctive fork, then (contrary to the claim in [1, p. 179])  $B$  need not be causally between  $A$  and  $C$  even if no two of  $A, B, C$  are  $P$ -equal: for example, if

$$\begin{aligned} P(ABC) &= 1/5, & P(AB\overline{C}) &= 1/5, \\ P(\overline{A}BC) &= 1/5, & P(\overline{A}B\overline{C}) &= 1/5, \\ P(A\overline{B}C) &= 0, & P(A\overline{B}\overline{C}) &= 0, \\ P(\overline{A}\overline{B}C) &= 0, & P(\overline{A}\overline{B}\overline{C}) &= 1/5, \end{aligned}$$

then  $(A, B, C)$  is a conjunctive fork and  $P(A|B) = P(A|C)$ .

If an event  $B$  is causally between  $A$  and  $C$ , then  $(A, B, C)$  need not be a conjunctive fork: for example, if

$$\begin{aligned} P(ABC) &= 1/20, & P(AB\overline{C}) &= 2/20, \\ P(\overline{A}BC) &= 2/20, & P(\overline{A}B\overline{C}) &= 4/20, \\ P(A\overline{B}C) &= 0, & P(A\overline{B}\overline{C}) &= 1/20, \\ P(\overline{A}\overline{B}C) &= 1/20, & P(\overline{A}\overline{B}\overline{C}) &= 9/20, \end{aligned}$$

then  $B$  is causally between  $A$  and  $C$  and  $P(AC|\overline{B}) \neq P(A|\overline{B})P(C|\overline{B})$ .

Following [3], we call a ternary relation  $\mathbf{b}$  on a finite ground set  $N$  an *abstract causal betweenness* if, and only if, there are events  $A_i$  with  $i$  ranging over  $N$  such that

$$\mathbf{b} = \{(i, j, k) \in N^3 : A_j \text{ is causally between } A_i \text{ and } A_k\}.$$

A natural question is which ternary relations  $\mathbf{b}$  form an abstract causal betweenness. This question was answered in [3, Theorem 1] in terms of the directed graph  $G(\mathbf{b})$  whose vertices are all two-element subsets of  $N$  and whose edges are all ordered pairs  $(\{i, j\}, \{i, k\})$  such that

$(i, j, k) \in \mathfrak{b}$  with  $i, j, k$  pairwise distinct:

- (36) a ternary relation  $\mathfrak{b}$  on a finite ground set  
is an abstract causal betweenness if and only if
- $(i, j, k) \in \mathfrak{b} \Rightarrow i, j, k$  are pairwise distinct,
  - $(i, j, k) \in \mathfrak{b} \Rightarrow (k, j, i) \in \mathfrak{b}$ ,
  - $G(\mathfrak{b})$  contains no directed cycle.

(The third requirement implies that  $(i, j, k) \in \mathfrak{b} \Rightarrow (i, k, j) \notin \mathfrak{b}$ : else  $G(\mathfrak{b})$  would contain the directed cycle  $\{i, j\} \rightarrow \{i, k\} \rightarrow \{i, j\}$ .)

An essential difference between abstract causal betweenness and fork-representable relations is that, on the one hand, every triple in an abstract causal betweenness consists of pairwise distinct elements and, on the other hand, a forkness includes with every triple  $(i, j, k)$  most of triples formed by at most two of  $i, j, k$ . This difference notwithstanding, the two can be compared. The trick is to introduce, for every ternary relation  $\mathfrak{r}$ , the ternary relation  $\mathfrak{r}^\#$  consisting of all triples in  $\mathfrak{r}$  that have pairwise distinct elements.

We claim that

- (37) if  $\mathfrak{r}$  is a fork-representable relation on a finite ground set  
such that  $(i, j, i) \notin \mathfrak{r}$  whenever  $i \neq j$ ,  
then  $\mathfrak{r}^\#$  is an abstract causal betweenness.

To justify this claim, consider a fork-representable relation  $\mathfrak{r}$  on a finite set  $N$  such that  $(i, j, i) \notin \mathfrak{r}$  whenever  $i \neq j$ . By Lemma 2,  $\mathfrak{r}$  is a forkness; assumption  $i \neq j \Rightarrow (i, j, i) \notin \mathfrak{r}$  implies that  $\sim$  is the identity relation, and so the quotient of  $\mathfrak{r}$  is isomorphic to  $\mathfrak{r}$ . Now Lemma 3 guarantees that  $\mathfrak{r}$  is solvable: there are positive numbers  $x_{\{i,j\}}$  such that  $x_{\{i,k\}} = x_{\{i,j\}} + x_{\{j,k\}}$  for all  $(i, j, k)$  in  $\mathfrak{r}$  with pairwise distinct  $i, j, k$ . Since  $x_{\{i,k\}} > x_{\{i,j\}}$  for every edge  $(\{i, j\}, \{i, k\})$  of  $G(\mathfrak{r}^\#)$ , this directed graph is acyclic, and so (36) guarantees that  $\mathfrak{r}^\#$  is an abstract causal betweenness.

Assumption  $i \neq j \Rightarrow (i, j, i) \notin \mathfrak{r}$  cannot be dropped from (37): consider  $\mathfrak{r} = N^3$ . This  $\mathfrak{r}$  is fork representable (for instance, by  $\Omega = \{x, y\}$ ,  $P(x) = P(y) = 1/2$ , and  $A_i = \{x\}$  for all  $i$  in  $N$ ). Nevertheless, if  $|N| \geq 3$ , then  $G(\mathfrak{r}^\#)$  contains cycles, and so  $\mathfrak{r}^\#$  is not an abstract causal betweenness.

The converse of (37),

if  $\mathfrak{r}^\sharp$  is an abstract causal betweenness  
 then  $\mathfrak{r}$  is a fork-representable relation  
 such that  $(i, j, i) \notin \mathfrak{r}$  whenever  $i \neq j$ ,

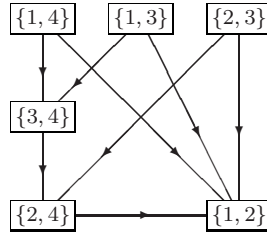
is false. Even its weaker version,

if  $\mathfrak{r}$  is a regular forkness  
 such that  $\mathfrak{r}^\sharp$  is an abstract causal betweenness,  
 then  $\mathfrak{r}$  is a fork-representable relation,

is false: consider the smallest forkness  $\mathfrak{r}$  on  $\{1, 2, 3, 4\}$  that contains the relation

$$\begin{aligned} & \{(1, 3, 2), (2, 3, 4), (3, 1, 4), (1, 4, 2), \\ & (2, 3, 1), (4, 3, 2), (4, 1, 3), (2, 4, 1)\}. \end{aligned}$$

Minimality of  $\mathfrak{r}$  implies that  $(i, j, i) \notin \mathfrak{r}$  whenever  $i \neq j$ ; it follows that  $\sim$  is the identity relation, and so  $\mathfrak{r}$  is a regular forkness. Graph  $G(\mathfrak{r}^\sharp)$  is acyclic



and so (36) guarantees that  $\mathfrak{r}^\sharp$  is an abstract causal betweenness. By Lemma 3,  $\mathfrak{r}$  is not fork representable: here, system (21) is isomorphic to

$$\begin{aligned} x_{\{1,2\}} &= x_{\{1,3\}} + x_{\{2,3\}} \\ x_{\{2,4\}} &= x_{\{2,3\}} + x_{\{3,4\}} \\ x_{\{3,4\}} &= x_{\{1,3\}} + x_{\{1,4\}} \\ x_{\{1,2\}} &= x_{\{1,4\}} + x_{\{2,4\}} \end{aligned}$$

and this system has no solution with  $x_{\{1,4\}} > 0$  as the linear combination of its four equations with multipliers  $-1, +1, +1, +1$  reads  $0 = 2x_{\{1,4\}}$ .

## 6. CONCLUDING REMARKS

1. The patterns studied in this work are based on combinations of conditional independence and covariance constraints for events. In recent decades, patterns of conditional independence among random variables have been studied in statistics and in probability theory since they provide insight to decompositions of multidimensional distributions, so sought for in applications. A framework for this activity was developed in the graphical models community[12].

A general formulation of the problem considers random variables  $\xi_i$  indexed by  $i$  in  $N$  and patterns consisting of the conditional independences  $\xi_i \perp\!\!\!\perp \xi_j | \xi_K$  where  $\xi_K = (\xi_k)_{k \in K}$ ,  $i, j \in N$ , and  $i, j \notin K$ . The case  $i = j$  means functional dependence of  $\xi_i$  on  $\xi_K$ , a.s. The problem is highly nontrivial even for four variables [15].

First treatments go back to [18, 23]. The variant of the problem excluding the functional dependence is most frequent [22]. Restrictions to Gaussian [13, 25] or binary variables, positivity of the distribution of  $\xi_N$ , etc., have been studied as well [4]. The idea to employ the Fourier-Stieltjes transform, as in Section 4, appeared in [14], characterizing patterns of unconditional independence.

2. For patterns of conditional independence, the role of forkness is played by graphoids [18], semigraphoids [17], imsets [22], semimatroids [15], etc. Notable are connections to matroid representations theory, see [16].

3. All possible patterns of conjunctive forks on events  $A_i$  indexed by  $i$  in  $N$  arise by varying a probability measure on  $\Omega$ , the power set of  $N$ . For a ternary relation  $\mathfrak{r}$  on a finite set  $N$ , the set  $\mathcal{P}_{\mathfrak{r}}$  of probability measures  $P$  on  $\Omega$  that satisfy  $(i, j, k) \in \mathfrak{r} \Leftrightarrow (A_i, A_j, A_k)_P$  is described by finitely many constraints that require quadratic polynomials in indeterminates  $z_{\omega}$  indexed by  $\omega$  in  $\Omega$  to be positive or zero. For fork-representability of  $\mathfrak{r}$ , it matters only whether  $\mathcal{P}_{\mathfrak{r}}$  is empty or not, which can be found out in polynomial time by the main result of the present paper. The shape of  $\mathcal{P}_{\mathfrak{r}}$ , which is a semialgebraic subset of the probability simplex, might be difficult to understand; to reveal it, finer algebraic techniques are needed, as in algebraic statistics [6, 26].

4. One of the two *Discrete Mathematics* reviewers asked: “Is there some interesting algebraic/combinatorial structure in admissible fork-nesses? Can they be partially ordered for a fixed  $N$ ?” We leave these questions open.

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## REFERENCES

- [1] P. von Bretzel (1977) Concerning a probabilistic theory of causation adequate for the causal theory of time. *Synthese* **35** 173–190.
- [2] N. Cartwright and M. Jones (1991) How to hunt quantum causes. *Erkenntnis* **35** 205–231.
- [3] V. Chvátal and B. Wu (2012) On Reichenbach’s causal betweenness. *Erkenntnis* **76** 41–48.
- [4] A.P. Dawid (1979) Conditional independence in statistical theory (with discussion). *J.R. Statist. Soc. B* **41** 1–31.
- [5] P. Dowe (1992) Process causality and asymmetry. *Erkenntnis* **37** 179–196.
- [6] M. Drton, B. Sturmfels and S. Sullivant (2009) *Lectures on Algebraic Statistics*. Oberwolfach Seminars, Springer Science & Business Media.
- [7] F.S. Ellett, Jr. and D.P. Ericson (1986) Correlation, partial correlation, and causation. *Synthese* **67** 157–173.
- [8] G. Hofer-Szabó, M. Rédei and L.E. Szabó (1999) On Reichenbach’s common-cause principle and Reichenbach’s notion of common cause. *The British Journal for the Philosophy of Science* **50** 377–399.
- [9] P.L. Kendall and P.F. Lazarsfeld (1950) Problems of survey analysis. In: *Continuities in Social Research: Studies in the Scope and Method of “The American Soldier”* (R.K. Merton and P.F. Lazarsfeld, eds.) The Free Press, Glencoe, IL, pp. 133–196.
- [10] L.G. Khachiyan (1979) A polynomial algorithm in linear programming. (in Russian) *Doklady Akademii Nauk SSSR* **244** 1093–1096.
- [11] K.B. Korb (1999) Probabilistic causal structure. In: *Causation and Laws of Nature* (H. Sankey, ed.), Kluwer Academic Publishers, Dordrecht, The Netherlands, pp. 265–311.
- [12] S.L. Lauritzen (1996) *Graphical Models*. Oxford University Press, Oxford.
- [13] R. Lněnička and F. Matúš (2007) On Gaussian conditional independence structures. *Kybernetika* **43** 327–342.



- [14] F. Matúš (1994) Stochastic independence, algebraic independence and abstract connectedness. *Theoretical Computer Science* **134** 455–471.
- [15] F. Matúš (1999) Conditional independences among four random variables III: final conclusion. *Combinatorics, Probability & Computing* **8** 269–276.
- [16] F. Matúš (1999) Matroid representations by partitions. *Discrete Mathematics* **203** 169–194.
- [17] F. Matúš (2004) Towards classification of semigraphoids. *Discrete Mathematics* **277** 115–145.
- [18] J. Pearl (1988) *Probabilistic Reasoning in Intelligent Systems*. Morgan Kaufman: San Mateo, California, 1988.
- [19] H. Reichenbach (1956) *The direction of time*. University of California Press, Berkeley and Los Angeles.
- [20] W.C. Salmon (1980) Probabilistic causality. *Pacific Philosophical Quarterly* **61** 50–74.
- [21] W.C. Salmon (1984) *Scientific Explanation and the Causal Structure of the World*. Princeton: Princeton University Press.
- [22] M. Studený (2005) *Probabilistic conditional independence structures*. Springer, New York.
- [23] W. Spohn (1980) Stochastic independence, causal independence, and shieldability. *Journal of Philosophical Logic* **9** 73–99.
- [24] W. Spohn (1994) On Reichenbach’s principle of the common cause. In: *Logic, Language, and the Structure of Scientific Theories* (W. Salmon and G. Wolters, eds.), University of Pittsburgh Press, Pittsburgh, pp. 211–235.
- [25] S. Sullivant (2009) Gaussian conditional independence has no finite complete axiom system. *Journal of Pure and Applied Algebra* **213** 1502–1506.
- [26] P. Zwiernik (2015) *Semialgebraic Statistics and Latent Tree Models*. CRC Press.

DEPARTMENT OF COMPUTER SCIENCE AND SOFTWARE ENGINEERING, CONCORDIA UNIVERSITY, MONTREAL, QUEBEC H3G 1M8, CANADA

*E-mail address*, V. Chvátal: [chvatal@cse.concordia.ca](mailto:chvatal@cse.concordia.ca)

INSTITUTE OF INFORMATION THEORY AND AUTOMATION, ACADEMY OF SCIENCES OF THE CZECH REPUBLIC, POD VODÁRENSKOU VĚŽÍ 4, 182 08 PRAGUE 8, CZECH REPUBLIC

*E-mail address*, F. Matúš: [matus@utia.cas.cz](mailto:matus@utia.cas.cz)

GOOGLE DEEPMIND, LONDON, UNITED KINGDOM

*E-mail address*, Y. Zwólš: [yori@google.com](mailto:yori@google.com)